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Solving the anharmonic oscillator problem with the *SU*(2) group

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Abstract

A new perspective for the anharmonic oscillator problem based on the $SU(2)$ group method (SGM) is provided. One finds that the SGM is a possible unified approach to treat both the harmonic oscillator and the anharmonic oscillator, although for the latter only part of the energy spectrum can be obtained. Coordinate translation $x \to x + \lambda$ for the anharmonic potential is also discussed, as one expects that the energy spectrum of an anharmonic oscillator is not affected by the translation.

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1. Introduction

One of the basic problems of non-relativistic quantum mechanics is to find the energy spectrum and eigenfunctions of a microsystem governed by the Schrödinger equation with an appropriate potential. Exact solutions of this equation are found for a limited class of potentials such as the harmonic oscillator (HO), the Coulomb potential and others. An anharmonic oscillator (AHO) has played an important role in the evolution of many branches of quantum mechanics. It serves as a basis for checking different approximate methods in quantum mechanics, the simplified counterpart of field-theoretical models, etc. Apart from this, it is of interest in its own right since the real world certainly deviates from the idealized picture of HOs due to the anharmonic interaction. The pioneering work of Simon [1] and Bender and Wu [2] has generated a vast amount of literature on AHOs. An overview of the work before 1980 can be found in the article by Killingbeck [3], and the current references can be found in [4–9].

In spite of its seeming simplicity, it is not an easy problem to find the energy spectrum and eigenfunctions of an anharmonic interaction. The standard way of solving this problem is to invoke perturbation theory. Perturbation series for any physical characteristics are typically asymptotic ones, i.e. they have a zeroth radius of convergence. Moreover, summation methods are required to account for higher order corrections. As a result such a method can be quite cumbersome. A thorough discussion of these difficulties has been given by Stevenson [10]. Indeed, numerous numerical methods, including renormalized strong-coupling expansion, renormalized perturbation expansion, supersymmetric quantum mechanics, WKB, iteration based on the generalized Bloch equation, state-dependent diagonalization, the Hill determinant method, the phase-integral approach, iterative Bogoliubov transformations, the eigenvalue moment method, the perturbative variation and the algebraic method, have been proposed to investigate the AHOs [11–27]. Recently, a general procedure based on shift operators has been formulated to deal with the AHO [9]. Under some definite consistency relations, analytic expressions for the exact ground-state energy can be derived for a large class of one-dimensional oscillators with cubic–quartic anharmonic potential $V(x) = \alpha x^2 + \beta x^3 + \gamma x^4$. The analytic results agree with the existing numerical methods, such as the method of state-dependent diagonalization and the divergent perturbation expansion. However, due to its inherent intractability, no further analytic solution for the energy spectrum and eigenstates has been obtained. Hence, under such a circumstance, it is appropriate to look for a new efficient method, so that new analytic solutions of energies and eigenstates can be obtained.

The aim of this paper is to provide a new perspective for the AHO based on the *SU*(2) group method (SGM). Strange as it may seem, to the best of our knowledge, the SGM has not been discussed in the literature in spite of the developed formalism relating realization of Lie algebras in Fock space and properties of differential equations. It is well known that generators $\{\hat{a}, \hat{a}^+, \hat{I}\}$ form the Heisenberg–Weyl group:

$$[\hat{a}, \hat{a}^+] = \hat{I} \qquad [\hat{a}, \hat{I}] = [\hat{a}^+, \hat{I}] = 0 \tag{1}$$

where $\hat{I}$ is the identity operator and $\hat{a}$ and $\hat{a}^+$ are annihilation and creation operators. The number operator can be constructed as $\hat{N} = \hat{a}^+\hat{a}$, whose eigenstates $|n\rangle$ $(n = 0, 1, \ldots)$ with corresponding eigenvalues n span the Fock space. Well known realization of $\hat{a}$ and $\hat{a}^+$ through x and $\mathrm{d}/\mathrm{d}x$ is

$$\hat{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{2}$$

In terms of these generators, the Hamiltonian of the harmonic oscillator $H = (-\mathrm{d}^2/\mathrm{d}x^2 + x^2)/2$ (in units of $\hbar = m = \omega = 1$) can be written as $H = \hat{a}^+\hat{a} + \hat{I}/2$. Expressed in this latter form, one may say that H has been factorized into shift operators $\hat{a}$ and $\hat{a}^+$. However, we would like to stress here that H can always be written in terms of the generators of the Heisenberg–Weyl algebra. Moreover, our ability to reduce it to a simpler form often results in better insight and understanding. Since the eigenvalues of $\hat{N}$ and $\hat{I}$ are known, the energy spectrum $E_n = n + 1/2$ of the HO can then be obtained immediately. Indeed, there are other kinds of realizations for $\hat{a}$ and $\hat{a}^+$, such as

$$\hat{a} = \frac{\mathrm{d}}{\mathrm{d}x} \qquad \hat{a}^+ = x. \tag{3}$$

For this case H can be written in terms of generators of Heisenberg–Weyl algebra as $H = [-\hat{a}^2 + (\hat{a}^+)^2]/2$, but such a form is generally not helpful for determining the energy spectrum. The above example of a HO tells us that, if we want to determine the energy spectrum of a general Hamiltonian $H \equiv H(x, \mathrm{d}/\mathrm{d}x) = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ by a group theoretic method [28], we should first select an appropriate *group* as well as a suitable realization of its generators formed by using x and $\mathrm{d}/\mathrm{d}x$. Further analysis shows that the Heisenberg–Weyl group and realizations of its generators in equation (2) which work well for a HO do not work well for an AHO. Thus, in order to deal with the eigenvalue problem of an AHO, we need to resort to other kinds of Lie groups. Naturally, a good candidate could be the $SU(2)$ group represented by

$$[j_+, j_-] = 2j_0 \qquad [j_0, j_\pm] = \pm j_\pm. \tag{4}$$

According to [29], the generators $j_\pm$ and j_0 are realized as

$$j_+ = 2j\xi - \xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi} \qquad j_0 = -j + \xi \frac{\mathrm{d}}{\mathrm{d}\xi} \qquad j_- = \frac{\mathrm{d}}{\mathrm{d}\xi} \tag{5}$$

which act on a space of polynomials of degree ξ^{2j}, i.e. the eigenfunction $\phi(\xi) = \sum_{m=0}^{2j} a_m \xi^m$, where j ($j = 0, 1/2, 1, \ldots$) is the spin (note that $j_-\xi^m = 0$ if $m = 0$, and $j_+\xi^m = 0$ if $m = 2j$).

Now for the Schrödinger equation

$$H\Psi = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\Psi = E\Psi \tag{6}$$

after performing the following transformation

$$\Psi(x) = \exp\left(-\int_0^x W(x)\,\mathrm{d}x\right)\psi(x) \tag{7}$$

we obtain from equation (6) that

$$H\psi = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2W\frac{\mathrm{d}}{\mathrm{d}x} - (W^2 - W' - V)\right)\psi = E\psi. \tag{8}$$

Transforming the variable x to ξ by $\xi = f(x)$ in the above equation, one then obtains

$$\begin{aligned} H\phi = \Bigg(&-f''(x)\frac{\mathrm{d}}{\mathrm{d}\xi} - [f'(x)]^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 2W(x)f'(x)\frac{\mathrm{d}}{\mathrm{d}\xi} \\ &- [W^2(x) - W'(x) - V(x)]\Bigg)\phi = E\phi. \end{aligned} \tag{9}$$

The next task is to write the H in equation (9) in terms of a combination of the $SU(2)$ generators, based on which, all or some of the energies and eigenfunctions can be constructed.

This paper is organized as follows. In section 2, we apply the SGM to the usual HO, so that one could understand better how the technique works for the AHO in section 3. Translation of the coordinate x for $V(x)$ is discussed in section 4. We end the paper with some relevant discussions in the last section.

2. Applying SGM to the usual harmonic oscillator

In this section, we apply the $SU(2)$ group method to the usual HO. Although the results in this case are well known, this simple model provides much insight into our technique and serves as a good instructive tool for the same formulation in the anharmonic case. The Hamiltonian of the HO is given by

$$H = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right) \qquad V(x) = x^2. \tag{10}$$

The function $W(x)$ is taken to be $W(x) = x$, which yields $W^2(x) - W(x) - V(x) = -1$. Let $\xi = f(x) = x$, so from equations (9) and (10) we find

$$H\phi = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 2\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + 1\right)\phi = E\phi \tag{11}$$

which can be written in terms of $j_\pm$ and j_0 as

$$H\phi = \frac{1}{2}\left[Aj_-^2 + Cj_0 + K\right]\phi = E\phi \qquad A = -1 \quad C = 2 \quad K = (1+2j). \tag{12}$$

Since $[H, \boldsymbol{j}^2] = 0$, where $\boldsymbol{j}$ is the spin angular momentum, we can choose $\phi(\xi) = \sum_{m=0}^{2j} a_m \xi^m$ to be common eigenfunctions of H and $\boldsymbol{j}^2$. For $j = 0$, it is easy to have $\phi_0(\xi) = 1$, so that the energy

$$E_0 = \frac{K}{2} = \frac{1}{2} \tag{13}$$

with the corresponding eigenfunction

$$\Psi_0(x) = \exp\left(-\int_0^x W(x)\,\mathrm{d}x\right) = \exp\left(-\frac{1}{2}x^2\right). \tag{14}$$

To determine the spectrum, we shall treat the eigenvalue problem corresponding to equation (12) as a matrix eigenvalue problem. To do this, we shall consider the $(2j+1)\times(2j+1)$ matrix representation of the operators $j_\pm$ and j_0. It is instructive to proceed inductively starting with $j = 1/2, 1, 3/2, \ldots$.

(i) $j = 1/2$. In this case,

$$j_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad j_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad j_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Substituting equation (15) into equation (12), one has

$$G\phi = 0 \qquad G = \begin{pmatrix} \frac{C}{2} + K - 2E & 0 \\ 0 & -\frac{C}{2} + K - 2E \end{pmatrix}. \tag{16}$$

Recognizing $\det G = 0$ immediately yields the energies

$$E_{j=1/2}^{-1/2} = \frac{1}{2} \qquad E_{j=1/2}^{1/2} = \frac{3}{2} \tag{17}$$

or $E_n = n + 1/2$ ($n = 0, 1, \ldots, 2j = 1$) with the corresponding eigenfunctions

$$\phi_{j=1/2}^{-1/2} = \begin{pmatrix} 0 \\ a_0 \end{pmatrix} \equiv a_0 \qquad \phi_{j=1/2}^{1/2} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \equiv a_1\xi. \tag{18}$$

Here we have used the notation $\phi(\xi) = (a_{2j}, a_{2j-1}, \ldots, a_1, a_0)^T \equiv \sum_{m=0}^{2j} a_m\xi^m$, where T denotes the transpose of an array.

(ii) $j = 1$. In this case,

$$j_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad j_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad j_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

After substituting equation (19) into equation (12), one has

$$G\phi = 0 \qquad G = \begin{pmatrix} C + K - 2E & 0 & 0 \\ 0 & K - 2E & 0 \\ 2A & 0 & -C + K - 2E \end{pmatrix}. \tag{20}$$

From $\det G = 0$, we have a cubic algebraic equation for the energy E:

$$(C + K - 2E)(K - 2E)(-C + K - 2E) = 0. \tag{21}$$

Solving the algebraic equation (21), we have

$$E_{j=1}^{-1} = \frac{1}{2}(K - C) = \frac{1}{2} \qquad E_{j=1}^{0} = \frac{1}{2}K = \frac{3}{2} \qquad E_{j=1}^{1} = \frac{1}{2}(K + C) = \frac{5}{2} \tag{22}$$

or $E_n = n + 1/2$ ($n = 0, 1, \ldots, 2j = 2$) with the eigenfunctions

$$\phi_{j=1}^{-1} = \begin{pmatrix} 0 \\ 0 \\ a_0 \end{pmatrix} \equiv a_0 \qquad \phi_{j=1}^{0} = \begin{pmatrix} 0 \\ a_1 \\ 0 \end{pmatrix} \equiv a_1\xi$$

$$\phi_{j=1}^{1} = \begin{pmatrix} Ca_0/A \\ 0 \\ a_0 \end{pmatrix} \equiv a_0(1 - 2\xi^2). \tag{23}$$

Actually, $\phi_{j=1}^{-1}$, $\phi_{j=1}^{0}$ and $\phi_{j=1}^{1}$ correspond to the first three respective Hermite polynomials. Interestingly, for the usual HO, it can be observed that $E_{j=1/2}^{-1/2} = E_{j=1}^{-1}$ and $E_{j=1/2}^{1/2} = E_{j=1}^{0}$ $\left(\text{or}\, \phi_{j=1/2}^{-1/2} = \phi_{j=1}^{-1} \text{ and } \phi_{j=1/2}^{1/2} = \phi_{j=1}^{0}\right)$.

(iii) For arbitrary j. Using

$$j_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle \qquad j_0|jm\rangle = m|j, m\rangle \tag{24}$$

the $(2j + 1) \times (2j + 1)$ matrix representations of $j_\pm$ and j_0 are

$$j_+ = \begin{pmatrix} 0 & \sqrt{1\cdot 2j} & 0 & 0 & & \cdots & \cdots \\ 0 & 0 & \sqrt{2\cdot(2j-1)} & 0 & & \cdots & \cdots \\ 0 & 0 & 0 & \sqrt{3\cdot(2j-2)} & & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{(2j-1)\cdot 2} & 0 \\ 0 & 0 & 0 & 0 & & 0 & \sqrt{2j\cdot 1} \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{pmatrix}$$

$$j_- = \begin{pmatrix} 0 & 0 & & 0 & 0 & 0 & 0 \\ \sqrt{1\cdot 2j} & 0 & & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2\cdot(2j-1)} & & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{(2j-2)\cdot 3} & 0 & 0 & 0 \\ 0 & 0 & & 0 & \sqrt{(2j-1)\cdot 2} & 0 & 0 \\ 0 & 0 & & 0 & 0 & \sqrt{2j\cdot 1} & 0 \end{pmatrix}$$

$$j_0 = \begin{pmatrix} j & & & & & & \\ & j-1 & & & & & \\ & & j-2 & & & & \\ & & & \ddots & & & \\ & & & & -j+2 & & \\ & & & & & -j+1 & \\ & & & & & & -j \end{pmatrix}. \tag{25}$$

Similarly, we obtain $E_n = n + \frac{1}{2}$ ($n = 0, 1, \ldots, 2j$), with the eigenfunctions ϕ_j^m ($m = -j, -j + 1, \ldots, j$) which correspond to the first $2j + 1$ Hermite polynomials. Also, one can find that the energies $E_{j'}^{m'}$ ($j' = j - 1,\ m = -j', -j' + 1, \ldots, j'$) are just the first $2j' + 1$ (=$2j - 1$) energies of E_j^m ($m = -j, -j + 1, \ldots, j$) this originates from the fact that the energy spectrum of a HO is equally spaced. One expects that the SGM is also applicable for an AHO. However, the finding, i.e. $E_{j'}^{m'}$'s are just the first $2j'+1$ energies of E_j^m's, will not be applicable for an AHO, since the energy spectrum of an AHO is generally not equally spaced.

In passing we would like to point out that, since the HO is the simplest model, there exists another simpler method to tackle the eigenvalue problem of equation (12), instead of adopting the $(2j+1)\times(2j+1)$ matrix representation of $j_\pm$ and j_0. The simpler method is to introduce a similarity transformation for equation (12) using the operator $U = \exp(-j_-^2/2)$, giving the Hamiltonian H'

$$H' = UHU^{-1} = \tfrac{1}{2}[Cj_0 + K]. \tag{26}$$

It is known that the eigenvalues of H' are the same as those of H under the similarity transformation. Since the eigenfunction of H' is just ξ^n $(n = 0, 1, 2, \ldots)$ with the corresponding eigenfunction

$$\Psi_n(x) = \exp\left(-\tfrac{1}{2}x^2\right)\exp\left(-\tfrac{1}{2}j_-^2\right)\xi^n. \tag{27}$$

The eigenvalue of j_0 is obtained to be $(n-j)$, thus eigenvalues of H' are

$$E'_n = \tfrac{1}{2}[(n-j)C + K] = n + \tfrac{1}{2} \quad (n = 0, 1, \ldots) \tag{28}$$

which are nothing but the energies of the usual HO.

3. Applying SGM to the anharmonic oscillator

In this section, we confine ourselves to the one-dimensional HO with the anharmonic potential containing cubic and quartic terms. However, the procedure developed here could be applied to some other types of anharmonic potentials. The Hamiltonian reads

$$H = -\frac{d^2}{dx^2} + V(x) \qquad V(x) = \alpha x^2 + \beta x^3 + \gamma x^4. \tag{29}$$

In this case, the corresponding function $W(x)$ for the above equation can be chosen as

$$W(x) = \mu x^2 + \tau x + \nu \tag{30}$$

so that the degree of x in the expression

$$\begin{aligned} W^2(x) - W'(x) - V(x) &= (\mu^2 - \gamma)x^4 + (2\mu\tau - \beta)x^3 + (\tau^2 + 2\mu\nu - \alpha)x^2 \\ &\quad + (2\tau\nu - 2\mu)x + \nu^2 - \tau \end{aligned} \tag{31}$$

is not greater than 2. In other words, we require that $\mu^2 - \gamma = 0$ and $2\mu\tau - \beta = 0$, which yield

$$\mu_1 = \sqrt{\gamma} \qquad \tau_1 = \frac{\beta}{2\sqrt{\gamma}} \qquad \text{or} \qquad \mu_2 = -\sqrt{\gamma} \qquad \tau_2 = -\frac{\beta}{2\sqrt{\gamma}}. \tag{32}$$

Correspondingly, we have

$$\begin{aligned} W_1(x) &= \mu_1 x^2 + \tau_1 x + \nu = \sqrt{\gamma}x^2 + \frac{\beta}{2\sqrt{\gamma}}x + \nu \\ W_2(x) &= \mu_2 x^2 + \tau_2 x + \nu = -\sqrt{\gamma}x^2 - \frac{\beta}{2\sqrt{\gamma}}x + \nu. \end{aligned} \tag{33}$$

For $W_1(x)$, we have

$$W_1^2(x) - W_1'(x) - V(x) = \left(\frac{\beta^2}{4\gamma} + 2\nu\sqrt{\gamma} - \alpha\right)x^2 + \left(\frac{\beta\nu}{\sqrt{\gamma}} - 2\sqrt{\gamma}\right)x + \nu^2 - \frac{\beta}{2\sqrt{\gamma}}. \tag{34}$$

Then from equation (8) we find

$$H\psi = \left(-\frac{d^2}{dx^2} + 2\left(\sqrt{\gamma}x^2 + \frac{\beta}{2\sqrt{\gamma}}x + \nu\right)\frac{d}{dx} - \left[\left(\frac{\beta^2}{4\gamma} + 2\nu\sqrt{\gamma} - \alpha\right)x^2 + \left(\frac{\beta\nu}{\sqrt{\gamma}} - 2\sqrt{\gamma}\right)x + \nu^2 - \frac{\beta}{2\sqrt{\gamma}}\right]\right)\psi = E\psi. \qquad (35)$$

Choosing $\xi = f(x) = x$, the above equation becomes

$$H\phi = \left(-\frac{d^2}{d\xi^2} + 2\left(\sqrt{\gamma}\xi^2 + \frac{\beta}{2\sqrt{\gamma}}\xi + \nu\right)\frac{d}{d\xi} - \left[\left(\frac{\beta^2}{4\gamma} + 2\nu\sqrt{\gamma} - \alpha\right)\xi^2 + \left(\frac{\beta\nu}{\sqrt{\gamma}} - 2\sqrt{\gamma}\right)\xi + \nu^2 - \frac{\beta}{2\sqrt{\gamma}}\right]\right)\phi = E\phi. \qquad (36)$$

Now in order to write H in terms of a combination of the $SU(2)$ generators $j_\pm$ and j_0, we set

$$H\phi = \left(Aj_-^2 + Bj_+ + Cj_0 + Dj_- + K\right)\phi = E\phi \qquad (37)$$

where A, B, C, D and K are constants to be determined. After substituting the expressions for $j_\pm$ and j_0 as given in equation (5) into equation (37), we obtain

$$H\phi = \left(A\frac{d^2}{d\xi^2} + (-B\xi^2 + C\xi + D)\frac{d}{d\xi} + 2jB\xi + K - jC\right)\phi = E\phi. \qquad (38)$$

Comparing equations (36) and (38) we find

$$A = -1 \qquad -B = 2\sqrt{\gamma} \qquad C = \frac{\beta}{\sqrt{\gamma}} \qquad D = 2\nu \qquad 2jB = -\frac{\beta\nu}{\sqrt{\gamma}} + 2\sqrt{\gamma}$$

$$K - jC = -\nu^2 + \frac{\beta}{2\sqrt{\gamma}} \qquad \frac{\beta^2}{4\gamma} + 2\sqrt{\gamma}\nu - \alpha = 0$$

giving the explicit solution

$$A = -1 \qquad B = -2\sqrt{\gamma} \qquad C = \frac{\beta}{\sqrt{\gamma}} \qquad D = 2\nu$$

$$K = \left(j + \frac{1}{2}\right)C - \nu^2 \qquad \nu = \frac{2\gamma}{\beta}(1 + 2j) \qquad (39)$$

subject to the constraint

$$\frac{\beta^2}{4\gamma} + 2\sqrt{\gamma}\nu - \alpha = 0 \qquad \text{or} \qquad \frac{\beta^2}{4\gamma} + 4\frac{\gamma^{3/2}}{\beta}(1 + 2j) - \alpha = 0. \qquad (40)$$

Obviously, when $j = 0$, the constraint condition becomes

$$\frac{\beta^2}{4\gamma} + 4\frac{\gamma^{3/2}}{\beta} - \alpha = 0 \qquad (41)$$

which is naturally simply the constraint given in equation (47) of [9].

Again, since $[H, \boldsymbol{j}^2] = 0$, $\phi(\xi) = \sum_{m=0}^{2j} a_m\xi^m$ could be the common eigenfunctions of H and $\boldsymbol{j}^2$. For $j = 0$, it is easy to have $\phi_0(\xi) = 1$, thus the energy is

$$E_0 = K = -\nu^2 + \frac{\beta}{2\sqrt{\gamma}} = -4\frac{\gamma^2}{\beta^2} + \frac{\beta}{2\sqrt{\gamma}} \qquad (42)$$

with the corresponding eigenfunction

$$\Psi_0(x) = \exp\left(-\int_0^x W_1(x)\,dx\right) = \exp\left[-\left(\frac{1}{3}\sqrt{\gamma}x^3 + \frac{1}{4}\frac{\beta}{\sqrt{\gamma}}x^2 + \nu x\right)\right]. \qquad (43)$$

As a wavefunction, $\Psi_0(x)$ should be normalizable, i.e. $\int_{-\infty}^{+\infty}|\Psi_0(x)|^2\,\mathrm{d}x$ = finite number. Therefore, in addition, those values of α, β and γ resulting in $\int_{-\infty}^{+\infty}|\Psi_0(x)|^2\,\mathrm{d}x \to \infty$ should be ruled out.

Now to determine the spectrum, we shall treat the eigenvalue problem corresponding to equation (37) as a matrix eigenvalue problem. As in section 2, it is instructive to work out the cases $j = 1/2, 1, 3/2, \ldots$ successively.

(i) $j = 1/2$. In this case, we have

$$G\phi = 0 \qquad G = \begin{pmatrix} \frac{C}{2} + K - E & B \\ D & -\frac{C}{2} + K - E \end{pmatrix}. \tag{44}$$

From $\det G = 0$, we have the energies

$$E_{j=1/2}^{-1/2} = K - \sqrt{\frac{C^2}{4} + BD} \qquad E_{j=1/2}^{1/2} = K + \sqrt{\frac{C^2}{4} + BD} \tag{45}$$

with the corresponding eigenfunctions

$$\phi_{j=1/2}^{-1/2} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \equiv a_0 + a_1\xi \qquad \phi_{j=1/2}^{1/2} = \begin{pmatrix} a_1' \\ a_0' \end{pmatrix} \equiv a_0' + a_1'\xi \tag{46}$$

where

$$a_0 = 1 \qquad a_1 = 1 + \frac{\frac{C}{2} - \sqrt{\frac{C^2}{4} + BD}}{D} \qquad a_0' = 1$$
$$a_1' = 1 + \frac{\frac{C}{2} + \sqrt{\frac{C^2}{4} + BD}}{D}. \tag{47}$$

(ii) $j = 1$. In this case, we have

$$G\phi = 0 \qquad G = \begin{pmatrix} C + K - E & \sqrt{2}B & 0 \\ \sqrt{2}D & K - E & \sqrt{2}B \\ 2A & \sqrt{2}D & -C + K - E \end{pmatrix}. \tag{48}$$

From $\det G = 0$, we have a cubic algebraic equation regarding energy E given by

$$(K - E)^3 + p(K - E) + q = 0 \qquad p = -(C^2 + 4BD) \qquad q = 4AB^2. \tag{49}$$

Provided $R = \frac{p^3}{27} + \frac{q^2}{4} < 0$, the solutions are

$$E_{j=1}^{-1} = K - r\cos u \qquad E_{j=1}^{0} = K - r\cos\left(u + \frac{2\pi}{3}\right)$$
$$E_{j=1}^{1} = K - r\cos\left(u + \frac{4\pi}{3}\right) \tag{50}$$

where

$$r = \left(-\frac{4p}{3}\right)^{1/2} \qquad u = \frac{1}{3}\cos^{-1}\left[-\frac{q}{2}\left(-\frac{p}{3}\right)^{-3/2}\right]. \tag{51}$$

When $R \geqslant 0$, equation (49) has only one real root

$$E = \left(-\frac{q}{2} + \sqrt{R}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{R}\right)^{1/3}. \tag{52}$$

(iii) For arbitrary j, from $\det G = 0$, we will have a $(2j+1)$-degree algebraic equation for energy E under a definite constraint as shown in equation (40). In general, using numerical computations, we could obtain $2j+1$ energies E_j^m's. However, as one can observe, $E_{j'}^{m'}(j' = j-1, m = -j', -j'+1, \ldots, j')$ are not the first $2j'+1$ energies of E_j^m $(m = -j, -j+1, \ldots, j)$, since the energy spectrum of an AHO is generally not equally spaced.

Similarly, for $W_2(x)$ one has

$$H\phi = \left(A' j_-^2 + B' j_+ + C' j_0 + D' j_- + K'\right)\phi = \tilde{E}\phi \tag{53}$$

where

$$\begin{aligned} &A' = -1 \qquad B' = 2\sqrt{\gamma} \qquad C' = -\frac{\beta}{\sqrt{\gamma}} \qquad D' = 2\nu \\ &K' = \left(j + \frac{1}{2}\right)C' - \nu^2 \qquad \nu = \frac{2\gamma}{\beta}(1+2j) \end{aligned} \tag{54}$$

with accompanying constraint

$$\frac{\beta^2}{4\gamma} - 2\sqrt{\gamma}\nu - \alpha = 0 \qquad \text{or} \qquad \frac{\beta^2}{4\gamma} - 4\frac{\gamma^{3/2}}{\beta}(1+2j) - \alpha = 0. \tag{55}$$

In particular, for $j = 0$, the constraint condition simplifies considerably as

$$\frac{\beta^2}{4\gamma} - 4\frac{\gamma^{3/2}}{\beta} - \alpha = 0 \tag{56}$$

which naturally derives the constraint in equation (44) of [9]. It is easy to have $\phi_0(\xi) = 1$, and the energy is

$$\tilde{E}_0 = K' = -\nu^2 - \frac{\beta}{2\sqrt{\gamma}} = -4\frac{\gamma^2}{\beta^2} - \frac{\beta}{2\sqrt{\gamma}} \tag{57}$$

with the corresponding eigenfunction

$$\tilde{\Psi}_0(x) = \exp\left(-\int_0^x W_2(x)\,\mathrm{d}x\right) = \exp\left[\frac{1}{3}\sqrt{\gamma}x^3 + \frac{1}{4}\frac{\beta}{\sqrt{\gamma}}x^2 - \nu x\right]. \tag{58}$$

The spectrum can be obtained by treating the eigenvalue problem in equation (53) as a $(2j+1)\times(2j+1)$-matrix eigenvalue problem. We find that the forms of eigenvalues and eigenfunctions are the same as those for the case with $W_1(x)$, simply by replacing $A, B, \ldots, K$ with $A', B', \ldots, K'$.

4. Translation in x for the potential $V(x) = \alpha x^2 + \beta x^3 + \gamma x^4$

In this section, we briefly consider the effects of a translation in the coordinate x for the anharmonic potential. For simplicity, we set $\gamma = 1$ and denote $V_1(x) = \alpha x^2 + \beta x^3 + x^4$. Using the transformation $x \to \rho x + \lambda$, we have

$$\begin{aligned} V_2(x) = V_1(\rho x + \lambda) &= \rho^4 x^4 + \rho^3(4\lambda + \beta)x^3 + \rho^2(6\lambda^2 + 3\beta\lambda + \alpha)x^2 \\ &\quad + \rho\lambda(4\lambda^2 + 3\beta\lambda + 2\alpha)x + \lambda^2(\lambda^2 + \beta\lambda + \alpha). \end{aligned} \tag{59}$$

From the physical point of view, when $\rho = 1$, the coordinate translation $x \to x + \lambda$ should not alter the energy spectrum of $H_1 = -\mathrm{d}^2/\mathrm{d}x^2 + V_1(x)$. For instance, if we choose $4\lambda + \beta = 0$, $6\lambda^2 + 3\beta\lambda + \alpha = 0$, i.e.

$$\lambda = -\frac{\beta}{4} \qquad \alpha = \frac{3\beta^2}{8} \tag{60}$$

equation (59) then becomes

$$V_2(x) = x^4 + gx + v \qquad g = -\frac{\beta^3}{16} \qquad v = 3\left(\frac{\beta}{4}\right)^4. \tag{61}$$

In the following, the SGM is applied to the AHO $H_2 = -\mathrm{d}^2/\mathrm{d}x^2 + V_2(x)$. We can show that the energy spectrum of H_2 is indeed the same as that of H_1 provided the constraint as shown in equation (40) (or equation (55)) is prescribed.

The function $W(x)$ corresponding to equation (61) can be expressed as $W(x) = ax^2 + b$ with our usual requirement that the degree of x in the function

$$W^2(x) - W'(x) - V_2(x) = (a^2 - 1)x^4 + 2abx^2 - (2a + g)x + b^2 - v \tag{62}$$

be less than or equal to two. Thus, $a = 1$ or $a = -1$ and we have $W_1(x) = x^2+b$, or $W_2(x) = -x^2 + b$. For $W_1(x)$, equation (63) leads to

$$H_2\psi = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(x^2 + b)\frac{\mathrm{d}}{\mathrm{d}x} - [2bx^2 - (2 + g)x + b^2 - v]\right)\psi = \epsilon\psi. \tag{63}$$

Using $\xi = f(x) = x$, the above equation becomes

$$H_2\phi = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 2\left(2j\xi - \xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\right) + 2b\frac{\mathrm{d}}{\mathrm{d}\xi} - 2b\xi^2 + (4j + 2 + g)\xi - b^2 + v\right)\phi = \epsilon\phi. \tag{64}$$

If we set $b = 0$ and $g = -2 - 4j$, then we can write H_2 as

$$H_2\phi = \left(-j_-^2 - 2j_+ + v\right)\phi = \epsilon\phi. \tag{65}$$

For spin $j = 0$ (in this case $g = -2$), it is easy to have $\phi_0(\xi) = 1$, thus the energy is $\epsilon_0 = v$. Substituting $\alpha = 3\beta^2/8$, β and $\gamma = 1$ into the constraint (41), one has $\beta = 2^{5/3}$. So it is easy to verify that $\epsilon_0 = E_0 = 3 \times 2^{-4/3}$, i.e. the ground-state energy is invariant under the translation $x \to x + \lambda$. The same conclusion is also valid for the cases with $j = 1/2$ and $j = 1$. A similar analysis can be made for $W_2(x)$, one could find that $\tilde{\epsilon}_0$ is identical to $\tilde{E}_0$.

5. Discussion

To summarize, we have provided a new perspective on the AHO problem using a technique based on the $SU(2)$ group theory. The SGM is found to be a possible unified approach for treating the HO and the AHO, and in the latter case, one can in fact obtain *part* of the energy spectrum. Such an approach is an example of the so-called quasi-exactly solvable models [30]. We have also discussed the effects on the energy spectrum due to a translation of coordinate x for the anharmonic potential. As one would expect, the energy spectrum of an AHO is not affected by the transformation $x \to x + \lambda$. Unlike the case in a HO, there are usually some constraint conditions for the coefficients α, β and γ in the anharmonic potential $V(x)$. If we rewrite equation (40) as

$$\frac{\alpha - \frac{\beta^2}{4\gamma}}{4\frac{\gamma^{3/2}}{\beta}} = (1 + 2j) = \text{positive integers} \tag{66}$$

then the constraint condition becomes a very interesting one since it seems to play a role much akin to a *quantization condition*.[1] However, a deeper meaning behind these constraints for an AHO is still under investigation.

[1] Note that this condition refers to part of the spectrum.

It is known that the local behaviour of most solvable potentials reduces to that of the HO or the Pöschl–Teller (PT) potential [31]. We would eventually envisage the possibility that the SGM is also applicable for the PT Hamiltonian: $H_{PT} = -\mathrm{d}^2/\mathrm{d}x^2 + [\nu(\nu-1)k^2/\cos^2(kx)]$. If setting $W(x) = k\nu\tan(kx)$ and $\xi = f(x) = \sin(kx)$, we will arrive at $H_{PT}\phi = k^2\big[-j_-^2 + (j_0 + j + \nu)^2\big]\phi = E\phi$. For $j = 0$, one has $\Psi_0(x) = \cos^{\nu}(kx)$ and $E_0 = k^2\nu^2$, which are simply the ground state and the corresponding energy, respectively.

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